Final Exam

Instructions:

- This is an open book exam; you are allowed to have our textbook, "Elementary Differential Geometry", by B. O'Neill, but no other documents.
- You should read through the whole exam before you answer the questions.
- You may use Lemmas and Theorems that are demonstrated in the textbook, but you may not use results that are stated in exercises, without first proving them.

Recall: On a geometric surface M:

- If V, and W are vector fields on M, then $[V,W]$ is a vector field on M defined by

$$[V,W][f] = V[W[f]] - W[V[f]],$$

for all $f \in C^\infty(M)$. If

$$\begin{aligned} \boldsymbol{x} : \mathbb{R}^2 \supset D &\to M \\ (u_1, u_2) &\mapsto \boldsymbol{x}(u_1, u_2) \end{aligned}$$

is a coordinate patch , and $V = \sum_i v_i \boldsymbol{x}_{u_i}$ and $W = \sum_i w_i \boldsymbol{x}_{u_i}$, then

$$[V,W] = \sum_i \big(V[w_i] - W[v_i]\big)\boldsymbol{x}_{u_i}.$$

- For any 1-form ϕ on M,

$$d\phi(V,W) = V[\phi(W)] - W[\phi(V)] - \phi([V,W]),$$

for all vector fields V and W on M.

- The *gradient* of a function $f \in C^\infty(M)$ is the unique vector field $\operatorname{grad} f$ such that $\langle \operatorname{grad} f, V\rangle = df(V)$ for all vector fields V on M.

Problem 1 (20 pts)
Let $\beta : I \to \mathbb{R}^3$ be a regular curve parameterised by arc-length, with torsion $\tau \neq 0$. Show that, if the image of β lies on a sphere centred at $c \in \mathbb{R}^3$, then the curvature κ satisfies $\kappa > 0$, and

$$\beta - c = -\rho \mathbf{N} - \rho' \sigma \mathbf{B},$$

where $\rho = \kappa^{-1}$, and $\sigma = \tau^{-1}$. Express the radius of the sphere in terms of ρ and σ.

Problem 2 (25 pts)
Consider the surface patch $\boldsymbol{x} : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$\boldsymbol{x}(u,v) = \left(u, v, \frac{u^2}{a^2} + \epsilon \frac{v^2}{b^2} \right),$$

where $\epsilon = \pm 1$, and $a, b \in \mathbb{R}$ are non-zero constants.

1. **(6 pts)** Show that $\boldsymbol{x}$ is regular.
2. **(6 pts)** Calculate the coefficients E, F, and G of the first fundamental form.
3. **(8 pts)** Calculate the coefficients L, M, and N of the second fundamental form.
4. **(5 pts)** Calculate the Gaussian curvature of this surface.

Problem 3 (20 pts)
Let M be a geometric surface, and E_1, E_2 a frame field on M. Show that

$$K = E_2[\omega_{12}(E_1)] - E_1[\omega_{12}(E_2)] - \omega_{12}(E_1)^2 - \omega_{12}(E_2)^2.$$

You will probably want to employ the structural equations, and the basis formulae.

Problem 4 (25 pts)
Suppose $D \subseteq \mathbb{R}^2$ is open. We define a geometric surface M to be the region D equipped with the inner product defined by

$$\langle \boldsymbol{v}, \boldsymbol{w} \rangle_p = \frac{\boldsymbol{v} \cdot \boldsymbol{w}}{h(p)^2},$$

where $h : D \to \mathbb{R}$ satisfies $h(p) \neq 0$ for all $p \in D$, and the dot product on the right hand side is the usual inner product on $\mathbb{R}^2$.

In this exercise, $D = \{(u,v) \in \mathbb{R}^2 \,|\, u^2 + v^2 < 4r^2$, for some $r > 0$, and

$$h(u,v) = 1 - \frac{u^2 + v^2}{4r^2}.$$

We can consider the identity map to be a coordinate patch $\boldsymbol{x} : D \to M$.

1. **(5 pts)** Construct a frame field $\{E_1, E_2\}$ on M. Express the vector fields E_i in terms of the vector fields $\boldsymbol{x}_u$, and $\boldsymbol{x}_v$.
2. **(5 pts)** Compute $[E_1, E_2]$ and express it in terms of the basis $\{E_i\}$.
3. **(5 pts)** Compute $\operatorname{grad} f$, where $f(u,v) = u + 2v$, and express it in terms of the basis $\{\boldsymbol{x}_u, \boldsymbol{x}_v\}$.
4. **(10 pts)** Compute the connection form ω_{12} associated to your frame field, and express it in terms of the basis $\{\theta_i\}$ dual to $\{E_i\}$.